Solutions

Exam 2
Chapter 2 and Sections 3.1-3.2

Answer the following questions. *You must show your work to receive full credit.*

1. (2 points each) Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, and suppose the universal set is $U = \{1, 2, \ldots, 9\}$. List all the elements in the following sets.

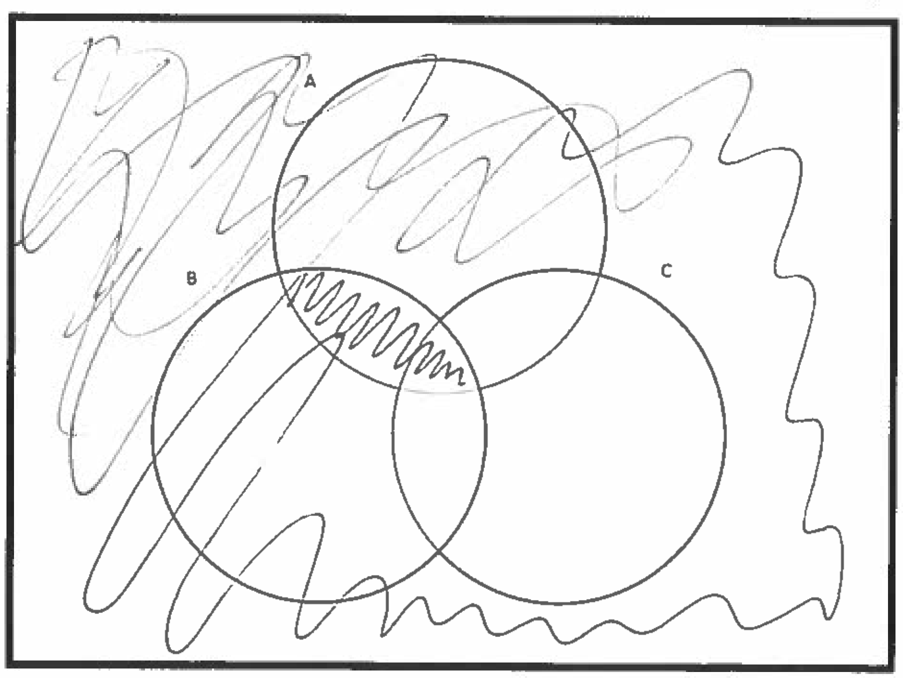
(a) $A' \cap B = \{5, 6\}$

(b) $(A \cup B)' = \{1, 7, 8, 9\}$

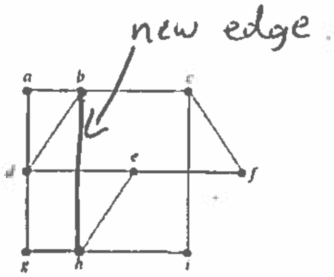
(c) $(A \cap B) \times A = \{(3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}$

(d) $P(B \setminus A) = \{\phi, \{5\}, \{6\}, \{5,6\}\}$

2. (8 points) In the venn diagram below, shade the area corresponding to $(A \cap B) \cup C'$.



3. Consider the following graph.



(a) (4 points) This graph does not have an Euler path (a path which uses every edge exactly once). Explain why. Make sure to reference any theorem that you use.

(b) (4 points) Without adding new vertices, add a single edge to the graph so that the new graph will have an Euler path. In indicate the new edge by drawing it on the graph.

(a) Vertices b, c, e, and h all have degree 3, so the Theorem of Euler tells us there is no Euler path.

(b) Connect any 2 of the vertice b, c, e and h and only 2 odd degree vertices. Then we will have an Euler path by the Theorem of Euler.

4. (2 points each) Consider the following sets. The universal set U for this problem is the set of all residents of India.

$A=$ the set of all English speakers. $B=$ the set of all Hindi speakers. $C=$ the set of all Urdu speakers.

Express the following sets in the symbols of set theory.

(a) Residents of India who speak English, Hindi, and Urdu.

(b) Residents of India who do not speak English, Hindi, or Urdu.

(c) Residents of India who speak English, but not Hindi or Urdu.

(a) $A \cap B \cap C$

(b) $(A \cup B \cup C)'$

(c) $A \cap (B \cup C)'$

5. (5 points) Let A and B be sets. Prove that $A \cap B \subseteq A \cup B$.

Let $x \in A \cap B$. Then $x \in A$. Therefore $x \in A \cup B$.

So $A \cap B \subseteq A \cup B$.

6. (3 points each) Define a function $f : \mathbb{R} \to \mathbb{R}$ by the formula $f(x) = 3x - 5$.

(a) Prove that f is one-to-one.

(b) Prove that f is onto.

(a) ~~Let $y \in \mathbb{R}$ then~~

Suppose $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$.

Then $3x_1 - 5 = 3x_2 - 5$

$$3x_1 = 3x_2$$

$x_1 = x_2$. So f is one-to-one.

(b) Let $y \in \mathbb{R}$. then, solving for the inverse, we have

$$y = 3x - 5$$
$$\frac{y+5}{3} = x.$$

$\Rightarrow$ Thus $f\left(\frac{y+5}{3}\right) = 3\left(\frac{y+5}{3}\right) - 5 = y$.

Therefore f is onto.

Let $n \in \mathbb{N}$ be a positive integer. Recall the relation on $\mathbb{Z}$ for modular arithmetic defined by $a \equiv b$ mod n whenever $n|(a-b)$.

7. (4 points) Find the equivalence class of 3 modular 10.

$[3] = \{n \in \mathbb{Z} \mid n = 10k+3 \text{ for some } k \in \mathbb{Z}\}$.

8. (9 points) Show that the above relation is an equivalence relation.

Reflexive: Let $a \in \mathbb{Z}$. Then $a-a=0=n \cdot 0$ so $a \equiv a \mod n$

Symmetric: Let $a,b \in \mathbb{Z}$ such that $a \equiv b \mod n$. Then $a-b=nk$ for some $k \in \mathbb{Z}$. Thus $b-a=-(a-b)=n\cdot(-k)$ and $b \equiv a \mod n$ ✓

Transitive: Let $a,b,c \in \mathbb{Z}$ such that $a \equiv b \mod n$ and $b \equiv c \mod n$. Then $a-b=nk_1$ and $b-c=nk_2$ for some $k_1,k_2 \in \mathbb{Z}$. Thus $a-c=(a-b)+(b-c)=nk_1+nk_2=n(k_1+k_2)$ and $n|a-c$. Therefore $a \equiv c \mod n$ ✓

~~**9.** (3 points) Show that the above relation is a partial ordering.~~

10. Let P be defined by

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + P(n-1) & \text{if } n > 1. \end{cases}$$

(a) (3 points) Use the recursive formula above to compute $P(4)$.

(b) (6 points) Use induction to verify that $P(n) = \frac{n(n+1)}{2}$.

(a) $P(1)=1,\ P(2)=3,\ P(3)=6,\ P(4)=10, \ldots$

(b) Base Case: $P(1) = \frac{1\cdot(1+1)}{2} = 1$ ✓

Inductive Step: Suppose $P(n) = \frac{n(n+1)}{2}$ for some $n \geq 1$.

Then $P(n+1) = (n+1) + P(n) = (n+1) + \frac{n(n+1)}{2}$

$$= \frac{2(n+1) + n(n+1)}{2} = \frac{(n+1)(n+2)}{2} \checkmark$$

Therefore $P(n) = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Extra Credit. Think of the internet as one big graph, where each web page is a vertex and each link is an edge.

(a) Is this a directed graph? Why or why not?

(b) Is this graph connected? Why or why not?

(c) Is this graph complete? Why or why not?

(d) Is this graph simple? Why or why not?

(e) For a given web page p, what does the outdegree of p represent?

(f) For a given web page p, what does the indegree of p represent?

(a) Yes, links only go one direction.

(b) Not necessarily. There are likely pages that never get linked to.

(c) Definitely not. No page links to all other pages.

(d) No, Some pages are self-referential or can link to the same page twice.

(e) The outdegree represents the # of links on the webpage.

(f) The indegree represents the number of links to p from other webpages.